SOME RESEARCH IN THE AREA OF THE DETERMINATION OF NONSTEADY HEAT FLUXES

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A method is proposed for the determination of intense heat fluxes on the basis of the solution of nonlinear equations of heat conduction with various boundary conditions.

Many technological problems presently require the determination of the analytical solution to nonlinear equations of heat conduction, permitting a deeper analysis of the thermal processes in the objects under consideration. From these solutions one can obtain suitable expressions for the determination of the heat fluxes delivered to the surfaces being heated.

This is the very question discussed in the present paper.

We note that in connection with the development of computational technology there are now no problems in obtaining a numerical solution for nonlinear equations of heat conduction. But the existing methods for the analytical solution of these equations are still distinguished by a certain complexity and awkwardness, by various degrees of approximation to the exact solution, and by a large amount of working time expended in computation. Therefore, the search for an analytically simple solution to the nonlinear equation of heat conduction is an urgent task and has considerable importance in the study of problems of heat exchange.

Let us consider an infinite copper plate with a thickness $R = 50 \cdot 10^{-3}$ m at an initial temperature:

$$t(x, \tau)|_{\tau=0} = t_0. \tag{1}$$

Let a heat flux

$$[\lambda_0 + \lambda_1 (t - t_0)] \left.\frac{\partial t}{\partial x}\right|_{x=0} = -q_0 (1 - e^{-\delta\tau}) = f(\tau) \tag{2}$$

act on the plane x = 0 of the plate at a time $\tau > 0$. In (2) it is assumed that $\lambda_0 = 390$ W/m·deg and $\lambda_1 = -0.0617$ W/m·deg^2, while $q_0 = 3 \cdot 10^7$ W/m^2 and $\delta = 31.54$ 1/sec are determined from the condition that the function $f(\tau)$ describe the experimental curve of [1], that being the most characteristic for the heating of copper heat-flux pickups by a high-temperature stream. Let the temperature of the opposite wall x = R of the plate remain constant during the entire heating process and consequently equal to

$$t(x, \tau)|_{x=R} = t_0. \tag{3}$$

From the solution of the nonlinear heat-conduction equation

$$\rho_0 [C_0 + C_1 (t - t_0)] \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left[(\lambda_0 + \lambda_1 (t - t_0)) \frac{\partial t}{\partial x}\right] \begin{pmatrix}\tau > 0, \\ 0 < x < R\end{pmatrix} \tag{4}$$

with the boundary conditions (1)-(3) we determine the temperature field of the plate as a function of time and of the spatial coordinate x, and then we compare it with the numerical solution of the same problem. For this we represent Eq. (4), in which we also take the values $\rho_0 = 8900$ kg/m^3, $C_0 = 387$ J/kg·deg, and $C_1 = 0.087$ J/kg·deg^2 for copper, in the form

$$\frac{\partial}{\partial \tau}\left(\Theta + \frac{C_1}{2C_0}\Theta^2\right) = a_0 \frac{\partial^2}{\partial x^2}\left(\Theta + \frac{\lambda_1}{2\lambda_0}\Theta^2\right), \tag{5}$$

where $\Theta(x, \tau) = t(x, \tau) - t_0$ while $a_0 = \lambda_0/\rho_0 C_0$ is the thermal diffusivity corresponding to the initial temperature t_0. Since the functions

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$$\varphi(\Theta) = \Theta + \frac{\lambda_1}{2\lambda_0}\Theta^2 \tag{6}$$

and

$$\psi(\Theta) = \Theta + \frac{C_1}{2C_0}\Theta^2 \tag{7}$$

are continuous in the interval $0 < \Theta < \Theta_{\Pi}$, they satisfy the Dirichlet conditions [2], and consequently they can be represented in the form of the following Fourier series:

$$\varphi(\Theta) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi\Theta}{\Theta_{\Pi}}, \tag{8}$$

$$\psi(\Theta) = \sum_{k=1}^{\infty} B_k \sin\frac{k\pi\Theta}{\Theta_{\Pi}}, \tag{9}$$

where

$$b_k = \frac{2}{\Theta_{\Pi}}\int_0^{\Theta_{\Pi}} \varphi(\Theta)\sin\frac{k\pi\Theta}{\Theta_{\Pi}}\,d\Theta \tag{10}$$

and

$$B_k = \frac{2}{\Theta_{\Pi}}\int_0^{\Theta_{\Pi}} \psi(\Theta)\sin\frac{k\pi\Theta}{\Theta_{\Pi}}\,d\Theta. \tag{11}$$

With allowance for (8) and (9) the system of equations (1)-(3) and (5) can be represented as

$$\frac{\partial}{\partial\tau}\sum_{k=1}^{\infty}\frac{B_k}{b_k}T_k + a_0\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty}T_k, \tag{12}$$

$$\sum_{k=1}^{\infty}T_k\Big|_{\tau=0} = 0, \tag{13}$$

$$\frac{\partial}{\partial x}\sum_{k=1}^{\infty}T_k\Big|_{x=0} = \frac{1}{\lambda_0 e}f(\tau)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}, \tag{14}$$

$$\sum_{k=1}^{\infty}T_k\Big|_{x=R} = 0 \tag{15}$$

or, after the formal application of the reduction rule [3], in the form

$$\frac{\partial T_k}{\partial\tau} = a_0\alpha_k\frac{\partial^2 T_k}{\partial x^2}, \tag{16}$$

$$T_k|_{\tau=0} = 0, \tag{17}$$

$$\frac{\partial T_k}{\partial x}\Big|_{x=0} = \frac{1}{\lambda_0 e}f(\tau)\frac{1}{(k-1)!}, \tag{18}$$

$$T_k|_{x=R} = 0, \tag{19}$$

where

$$T_k = b_k\sin\frac{k\pi\Theta}{\Theta_{\Pi}}, \tag{20}$$

e is the base of the natural logarithms, and

$$\alpha_k = \frac{b_k}{B_k} = \frac{\cos k\pi + \frac{\lambda_1}{2\lambda_0}\Theta_\Pi \left[\left(1 - \frac{2}{(k\pi)^2}\right)\cos k\pi + \frac{2}{(k\pi)^2}\right]}{\cos k\pi + \frac{C_1}{2C_0}\Theta_\Pi \left[\left(1 - \frac{2}{(k\pi)^2}\right)\cos k\pi + \frac{2}{(k\pi)^2}\right]}. \quad (21)$$

The solution of the system of equations (16)-(19) presents no difficulties. Thus, after the application of a Laplace transform with a subsequent transition to the inverse transform and summation over k, the solution has the form

$$\Theta(x, \tau) = -\frac{\lambda_0}{\lambda_1} + \frac{\lambda_0}{\lambda_1}\left[1 + 2\frac{\lambda_1}{\lambda_0}\Phi_1(x, \tau)\right]^{\frac{1}{2}}, \quad (22)$$

where

$$\Phi_1(x, \tau) = \frac{q_0}{\lambda_0}(R - x) - \frac{q_0}{\lambda_0 e}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\sqrt{\frac{a_0\alpha_k}{\delta}} \times \mathrm{sc}\sqrt{\frac{\delta}{a_0\alpha_k}}\sin\sqrt{\frac{\delta}{a_0\alpha_k}}(R - x)\exp[-\delta\tau] + \frac{2q_0R^3\delta}{e\lambda_0} \times \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\mu_n\frac{R-x}{R}}{(k-1)!\,\mu_n^2(a_0\alpha_k\mu_n^2 - \delta R^2)}\exp\left[-\frac{a_0\alpha_k\mu_n^2}{R^2}\tau\right], \quad (23)$$

while μ_n are the roots of the characteristic equation $\cos\mu_n = 0$.

Since the maximum temperature did not exceed 483.28°C in the numerical solution of the system of equations (1)-(4), in the determination of the temperature field from Eq. (22) the values of α_k can be taken from Table 1 with $\Theta_\Pi = 600$°C. Moreover, it follows from the table presented that α_1 differs from all the subsequent α_k by an insignificant amount. Consequently, it seems possible to simplify the solution (22) still more if out of all the α_k with $\Theta_\Pi = 600$°C one takes their mean value $\alpha_0 = 0.91357$. Then in the new solution (22) the expression (23) obtains the form

$$\Phi_2(x, \tau) = \frac{q_0}{\lambda_0}(R - x) - \frac{q_0}{\lambda_0}\sqrt{\frac{a_0\alpha_0}{\delta}} \times \mathrm{sc}\sqrt{\frac{\delta}{a_0\alpha_0}}\sin\sqrt{\frac{\delta}{a_0\alpha_0}}(R - x)\exp(-\delta\tau) + \frac{2q_0R^3\delta}{\lambda_0} \times \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\mu_n\frac{R-x}{R}}{\mu_n^2(a_0\alpha_0\mu_n^2 - \delta R^2)}\exp\left[-\frac{a_0\alpha_0\mu_n^2}{R^2}\right]. \quad (24)$$

In Table 2 we present a comparison of the temperatures for this very solution (top row) and for the numerical solution at different points of the plate as a function of time. As follows from the table, the absolute difference between them is small, and in this sense the solution (22) and the numerical solution can be considered as identical to a certain extent.

The proposed method of solving nonlinear equations of heat conduction allows one to obtain convenient equations for the determination of the heat fluxes reaching the surfaces being heated. Thus, by solving the heat-conduction equation (5) by the method presented, with boundary conditions of the form

$$\Theta|_{\tau=0} = 0, \quad (25)$$

$$\Theta|_{x=R_1} = \varphi_1(\tau), \quad (26)$$

$$\Theta|_{x=R_2} = \varphi_2(\tau), \quad (27)$$

after some elementary transformations we find that

$$q(\tau) = \lambda_0\left[(\psi_1(\tau) - \psi_2(\tau))\frac{1}{(1-\beta)R_2} + \Phi_1(\tau) + \Phi_2(\tau)\right], \quad (28)$$

TABLE 1. Values of the Coefficients α_k

k	θ_Π				
	200	400	600	800	1000
1	0,97754	0,95563	0,93434	0,91348	0,89324
2	0,96255	0,92669	0,89236	0,85944	0,82785
3	0,96422	0,92986	0,89694	0,86532	0,83483
4	0,96255	0,92669	0,89236	0,85939	0,82785
5	0,96317	0,92779	0,89394	0,86155	0,83032
6	0,96255	0,92669	0,89236	0,85939	0,82785
7	0,96282	0,92724	0,89314	0,86049	0,82910
8	0,96255	0,92669	0,89236	0,85939	0,82785
9	0,96273	0,92701	0,89289	0,86006	0,82862
10	0,96255	0,92669	0,89236	0,85939	0,82785
11	0,96268	0,92692	0,89271	0,85990	0,82834
12	0,96255	0,92669	0,89236	0,85939	0,82785
13	0,96259	0,92688	0,89262	0,85973	0,82822
14	0,96255	0,92669	0,89236	0,85939	0,82785
15	0,96259	0,92678	0,89253	0,85969	0,82814
16	0,96255	0,92669	0,89236	0,85939	0,82785
17	0,96259	0,92678	0,89245	0,85961	0,82805
18	0,96255	0,92669	0,89236	0,85939	0,82785
19	0,96259	0,92678	0,89244	0,85959	0,82804
20	0,96255	0,92669	0,89236	0,85939	0,82785

TABLE 2. Dependence of Temperature Field of Plate on Spatial and Time Coordinates

τ	x, mm					
	0	1	2	3	5	50
0,045	107,81	60,11	31,24	15,08	2,78	0
	112,27	64,25	34,58	17,45	3,61	0
0,065	156,15	98,16	58,72	33,36	9,15	0
	161,98	103,77	63,69	37,37	11,15	0
0,085	198,64	134,34	87,43	54,68	18,88	0
	205,39	140,94	93,58	60,04	22,16	0
0,105	235,98	167,72	115,54	77,07	30,99	0
	243,31	174,97	122,51	83,44	35,45	0
0,125	269,16	198,32	142,36	99,46	44,61	0
	276,84	205,96	149,86	106,55	50,07	0
0,145	299,08	226,48	167,71	121,34	59,08	0
	306,93	234,32	175,54	128,93	65,35	0
0,165	326,42	252,55	191,63	142,51	73,97	0
	334,31	260,47	199,63	150,40	80,88	0
0,185	351,72	276,88	214,27	162,88	89,01	0
	359,55	284,77	222,30	170,95	96,40	0
0,205	375,37	299,76	235,75	182,49	104,02	0
	383,07	307,53	243,74	190,62	111,76	0
0,225	397,67	321,41	256,23	201,37	118,90	0
	405,16	328,99	264,09	209,47	126,88	0
0,245	418,85	342,02	275,83	219,58	133,59	0
	426,07	349,35	283,49	227,57	141,70	0
0,265	439,06	361,73	294,65	237,18	148,06	0
	445,97	368,76	302,06	244,99	156,22	0
0,285	458,45	380,66	312,78	254,22	162,28	0
	465,00	387,35	319,89	261,81	170,43	0
0,305	477,12	398,90	330,30	270,76	176,27	0
	483,28	405,21	337,07	278,08	184,34	0

where

$$\Phi_1(\tau)=\psi_1'(\tau)\frac{(2+2\beta-\beta^2)R_2}{6a_0e(1-\beta)}\left(\frac{1}{\alpha_1}+\frac{e-1}{\alpha_k}\right),$$

$$\Phi_2(\tau)=\psi_2'(\tau)\frac{(1-2\beta-2\beta^2)R_2}{6a_0e(1-\beta)}\left(\frac{1}{\alpha_1}+\frac{e-1}{\alpha_k}\right),$$

$$\psi_1(\tau)=\varphi_1(\tau)+\frac{\lambda_1}{2\lambda_0}\varphi_1^2(\tau),$$

$$\psi_2(\tau)=\varphi_2(\tau)+\frac{\lambda_1}{2\lambda_0}\varphi_2^2(\tau),$$

$\beta = R_1/2$, $\alpha_1 = 0.93434$, and $\alpha_k = 0.89280$ is the arithmetic mean of all α_k but the first. In the given case the data needed for the functions $\varphi_1(\tau)$ and $\varphi_2(\tau)$ can be taken from the solution (22). In practical investigations they must be taken from the experiment at the points where the temperature sensors are mounted.

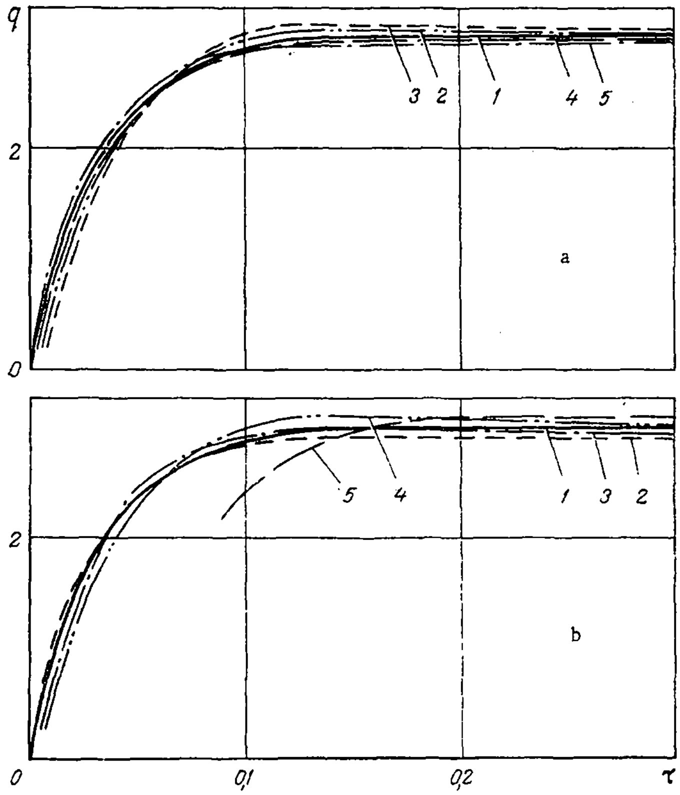

Fig. 1. Results of reconstruction of the heat flux [1: original dependence (2)]: for a) 2, 3, 4, and 5 from Eq. (28) with $R_1 = 1$ mm and $R_2 = 5$ mm; 2 and 5, 1 and 3, and 3 and 5 mm, respectively; for b) 2, 3, 4, and 5 from Eq. (30) at a distance R = 1, 2, 3, and 5 mm from heating surface. q, kW/cm^2; τ, sec.

The values of the heat fluxes obtained from Eq. (28) as a function of the location of the thermocouples are presented in Fig. 1a, from which it follows that these data are sufficiently close to the values of Eq. (2). This confirms the fact that the proposed method of solving nonlinear equations of heat conduction assures a high degree of accuracy.

In a number of practical cases it seems sufficient to mount only one rather than two thermocouples in the heat-flux pickups. Such a situation can arise when a body is heated over some time interval to an insignificant depth in comparison with its length. In this case the body can be taken as semiinfinite and the condition at infinity should be taken as the second boundary condition, i.e.,

$$\Theta|_{x=\infty} = 0. \tag{29}$$

Then, solving Eq. (5) with the boundary conditions (1), (26), and (29) by a method analogous to that of the preceding problem, after some simplifications we obtain an equation of the following form for the determination of the heat fluxes:

$$q(\tau) = \lambda_0 \left[\frac{1}{e\sqrt{\pi a_0 \alpha_1 \tau}} \left(\psi_1(\tau) \exp\left[-\frac{R_1^2}{4a_0\alpha_1\tau} \right] + F_1(\tau) \right) + \frac{e-1}{e\sqrt{\pi a_0 \alpha_k \tau}} \left(\psi_1(\tau) \exp\left[-\frac{R_1^2}{4a_0\alpha_k\tau} \right] + F_2(\tau) \right) \right], \tag{30}$$

where

$$F_1(\tau) = \psi_1'(\tau)\,\tau \left\{ \exp\left[-\frac{R_1^2}{4a_0\alpha_1\tau} \right] + R_1 \sqrt{\frac{\pi}{a_0\alpha_1\tau}} \left(1 + \operatorname{erf} \frac{R_1}{2\sqrt{a_0\alpha_1\tau}} \right) \right\}$$

and

$$F_2(\tau) = \psi_1'(\tau)\tau \left\{\exp\left[-\frac{R_1^2}{4a_0\bar{\alpha}_k\tau}\right] + R_1\sqrt{\frac{\pi}{a_0\bar{\alpha}_k\tau}}\left(1 + \operatorname{erf}\frac{R_1}{2\sqrt{a_0\bar{\alpha}_k\tau}}\right)\right\}.$$

The results of the reconstruction of the heat fluxes of (2) obtained from Eq. (30) are presented in Fig. 1b, as a function of the time and the position of the temperature sensor. It follows from the figure that placing the temperature sensor farther from the surface of the heat-flux pickup being heated leads to greater errors. This obviously follows from the violation of the approximation of the temperature field at a point close to the surface.

Thus, the most reliable results of q can be obtained in the case when the temperature measurement is made in the immediate vicinity of the heating surface.

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SPLINE IDENTIFICATION OF HEAT FLUXES

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A method is discussed for the determination of one-dimensional transient heat fluxes from the experimentally measured temperature using a spline approximation of the heat flux with subsequent application of the procedures of parametric identification.

A thermal experiment can be treated as a certain measuring system with an unknown input, subject to determination, and an output which is measured with noise. A one-dimensional body of finite length with known thermophysical characteristics, dependent on the temperature in the general case, with a thermally insulated lateral surface and with the temperature of the end being measured, serves as the physical model of a measuring system for the determination of a one-dimensional heat flux. Serving as the mathematical model for the measuring system is a system of equations consisting of a differential—difference system of equations, approximating the one dimensional Fourier heat-conduction equation by spatial quantization at n points, and the observation equation:

$$\left|\begin{aligned}&\dot{\mathbf{T}} = A\mathbf{T} + B\mathbf{Q},\\ &\mathbf{Y} = H\mathbf{T} + \mathbf{W},\end{aligned}\right. \qquad (1)$$

where

$$\mathbf{T} = [T_1\ T_2\ \ldots\ T_n]^t,\quad \mathbf{Q} = [q_1\ 0\ \ldots\ 0\ q_2]^t,\quad H = [1\ 0\ \ldots\ 0],\qquad B = \begin{bmatrix}\frac{1}{(c\rho)_1 h} & 0\\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\\ 0 & 0\\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\\ 0 & \frac{1}{(c\rho)_n h}\end{bmatrix},$$

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